Advanced Mathematical Models & Applications
Vol.8, No.3, 2023, pp.529-535

THE PROBLEM OF STABILITY OF A RECTANGULAR ARCH WITH RIGID PINCHING OF BOTH ENDS

L. Fatullayeva*, N. Fomina, N. Mammadova

Department of Mathematical Methods of Applied Analysis, Baku State University, Baku, Azerbaijan

Abstract. In the presented work, the variation method is applied to the solution of the stability problem of a rectangular arch with non-uniform thickness and stiff ends. The main goal here is to get expressions of functionality and critical force in the appropriate form necessary for solving the problem. In the article, a study of the stress-strain state of a rectangular arch in a geometric nonlinear formulation is given. For numerical calculations, it is assumed that the thickness of a rectangular arch consists of three layers and has a periodic structure.

Keywords: A variation method, the approximation argument, the boundary conditions, the critical force, a thickness, the Rayleigh-Ritz method.

AMS Subject Classification: 70G75, 70K20, 74-06.

Corresponding author: L. Fatullayeva, Department of Mathematical Methods of Applied Analysis, Baku State University, Z. Khalilov 23, AZ1148, Baku, Azerbaijan, e-mail: *laurafatullayeva@bsu.edu.az*

Received: 23 June 2023; Revised: 19 September 2023; Accepted: 13 October 2023;
Published: 20 December 2023.

1 Introduction

The science "Mechanics of a deformable solid body" studies engineering methods for calculating the elements of structures and machine parts for strength, rigidity and stability. The most universal and commonly used is the calculation of strength. Structural parts correctly calculated for strength work within the limits of elastic deformations. Only some particularly elastic parts and structural elements are subjected to stiffness and stability calculations, for example, long shafts, thin rods, multi-span beams and arches operating in vibration modes. To conclude that a structure is safe in the process of designing and operating buildings and engineering structures, conventional strength calculations are not enough: it is also necessary to take into account the rigidity and stability of structures when solving problems of material resistance. The reliability of structures is ensured by the correct pairing of structures with the joint correct operation of the elements. A combination of various reasons leads to the emergency state of structures: design errors, violation of the technology of manufacturing and installation of building structures, poor quality of materials used for load-bearing structures, non-compliance with the rules for the operation of buildings and structures - as a result, violation of the integrity of the frame of a building or structure. Emergencies are often accompanied by injuries, death of people and cause significant economic damage. Structural failures rarely occur suddenly. In almost all cases, a number of prerequisites for what happened can be traced. If you notice the signs in a timely manner, it is possible to take preventive measures in time, thereby preserving not only material values, but also the life and health of people (Todchuk, 2022; Wu et al., 2022).

How to cite (APA): Fatullayeva, L., Fomina, N. & Mammadova, N. (2023). The problem of stability of a rectangular arch with rigid pinching of both ends. *Advanced Mathematical Models & Applications*, *8*(3), 529-535.

The creation of composite materials and their widespread use in many areas of industry creates the need to determine the stress-strain state of the material body (the stress-strain state is a set of stresses and strains that arise when external loads, temperature fields, and other factors act on a material body) made from various nonlinear elastic materials. Parts of the material body, by their nature consisting of different materials, come into contact either by completely merging with each other, or by landing of each other. Landing is a parameter that characterizes the connection of parts of the material body. It is determined by the magnitude of the gaps or interferences resulting from the connection. If the body has a complex spatial configuration, if the deformations are finite, if an external pressure acts on the body, then mathematical difficulties arise for the analysis of the stress-strain state. This is due to the fact that theoretical studies carried out in this area lead to the integration of a system of nonlinear differential equilibrium equations. In addition, if the coefficients in the differential equations are discontinuous functions and the boundary conditions are non-linear, then this makes the solution of the problem even more difficult. Therefore, it is important to apply approximate methods, and in our case, the variation method, to solve such problems.

In Dzyuba et al. (2022) are given the results of a systematic experimental study of the subcritical behavior, supercritical forms of equilibrium and critical loads of cylindrical shells with a rectangular cutout of the lateral surface during transverse bending. The hole was located in the compressed zone of the shell. Five series of models (243 shells) were tested, for which the area and aspect ratio of the rectangular hole varied within wide limits. Models for research were made from special drawing paper, which has stable mechanical characteristics and high manufacturability of models. It is shown that despite the relatively low values of the elastic characteristics of this material in comparison with metals, the ratio of its yield strength to the modulus of elasticity is commensurate with these data for sheet stainless steel, which makes it possible to carry out large-scale experimental studies that are practically impossible to carry out on metal shells. It is indicated that the loss of stability of a cylindrical shell under conditions of transverse bending has its own characteristics due to the inhomogeneous stress state of the shell, both along the length and in the circumferential direction.

In aerospace systems and automotive industry flexible elastic and thin-walled structures are widely used (Dmitriev et al., 2020). The adequate mathematical models are in the process of development as well as numerical algorithms for examination of the features of processes of deformation of panel and arch designs with large displacements and arbitrary angles of rotation of the normal. For the discretization of the original continuum problem in spatial variables, the finite difference method can be used with the replacement of differential operators by finite-difference second-order approximations. The computational algorithm for solving a substantially nonlinear boundary-value problem is constructed on the basis of adaptation of the calculation method with construction of an explicit two-layer difference scheme in time of the second order of accuracy. The influence of the boundary conditions on the features of subcritical and supercritical behavior of the elastic arch structure under the influence of the surface load of the "tracking" type is considered.

In Emami et al. (2022) investigated the cyclic performance of arched steel haunches as a new strategy in the seismic retrofitting of reinforced concrete frames and focused on the slenderness ratio effect. A series of cyclic loading were conducted on six test specimens in two groups with the same nominal length and different axial eccentricities of 0.1 and 0.2 nominal length and with out-of-plane slenderness ratios of 138, 69, and 16. Experimental results indicated that the slenderness ratio played a very important role on cyclic performance in compression and even tension, so that a more desirable hysteretic behavior was achieved when the overall buckling potential was restricted. Therefore, by reducing this ratio, the maximum compressive and tensile strengths increased up to 1.78 and 1.28 times, respectively, and also the dissipated energy and maximum viscous damping upgraded up to 3.32 and 1.43 times, respectively. More difference in tensile and compressive behavior for ultimate strength and plastic stiffness was observed, when the initial eccentricity decreased.

In Fatullayeva et al. (2023) the effectiveness of the variation method proposed for the solution of the considered problem is shown on the problem of determining the stability of a rectangular arch whose ends are closed in different ways. The arch under consideration is under vertical pressure with an intensity that is regularly distributed along its surface. In the presented work, the effect of the geometric and physical parameters of the rectangular arch, which is the research object, on the value of the breaking force was investigated.

2 Functional to the considered problem

Assume that the arch under consideration is under a vertical pressure of intensity q uniformly distributed along its surface. The axis of a rectangular arch with both ends rigidly closed is given by the following formula:

$$w = c_0 \eta \sin \frac{\pi z}{l} \sin \pi \left(1 - \frac{z}{l}\right), \tag{1}$$

where c_0 is the axis of elevation of the arch, η is the approximation argument, l is the distance between the supports of the arch, y is the horizontal coordinate, and z vertical coordinate (Fig. 1).

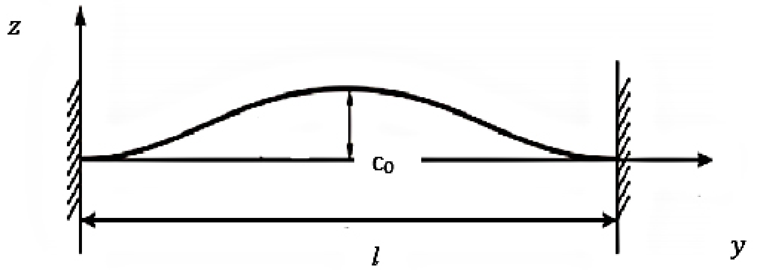

Figure 1: A model of a rectangular arch with both ends rigidly closed.

Expression (1) satisfies the boundary conditions of rigid closure of both ends, that is:

$$w(0) = w(l) = 0\,; \quad w_{,z}(0) = w_{,z}(l) = 0\,.$$

The cross-section of the arch is rectangular, its height is $2h$, and its width is b. It is assumed that the arch is geometrically non-linear, that is, it consists of a number n of layers with different thicknesses. Let us denote the thickness of each layer by δ_{k+1}, then succeed the following equality:

$$\sum_{k=0}^{n-1} \delta_{k+1} = 2h.$$

The functional is taken as follows (Fatullayeva et al., 2023):

$$J = b \int_{-h}^{h} \int_{0}^{l} \left\{ \dot{\sigma} \dot{\varepsilon} + \frac{1}{2} \sigma \dot{\omega}_{,z}^{2} \right\} dydz - \frac{b}{2} \int_{0}^{l} \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \frac{\dot{\sigma}^2}{E_{k+1}(y)} dydz + \int_{0}^{l} \dot{q} \dot{\omega} dz\,, \tag{2}$$

where σ is the stress, $E_{k+1} = E_{k+1}(y)$ ($k = 0,\, 1,\, ...,\, (s-1)$) is the modulus of elasticity of the material of the k-th layer, and a_k are defined in this way:

$$a_k = -h + \sum_{j=0}^{k} \delta_j \quad (\delta_0 = 0), a_k \leq y \leq a_{k+1}.$$

The rate of deformation ($\dot{\varepsilon}$) is defined as in work (Fatullayeva et al., 2023):

$$\dot{\varepsilon} = \omega_{,z}\dot{\omega}_{,z} - y\dot{\omega}_{,zz}. \tag{3}$$

In formulas (2) and (3) and below, the comma means partial differentiation with respect to the coordinate z, and the point means differentiation with respect to q, i.e. $\dot{q} = 1$.

The approximating function for σ and its speed are defined in the following form:

$$\sigma = E_1 \left(\sigma_0 \sin\left(\frac{\pi z}{l}\right) + \sigma_1 \sin\left(\frac{\pi z}{l}\right) \cdot \left(\frac{2y}{h}\right) \right), \tag{4}$$

$$\dot{\sigma} = E_1 \left(\dot{\sigma}_0 \sin\left(\frac{\pi z}{l}\right) + \dot{\sigma}_1 \sin\left(\frac{\pi z}{l}\right) \cdot \left(\frac{2y}{h}\right) \right). \tag{5}$$

To obtain the final formula for this functional, we substitute formulas (3)-(5) into (2) and perform the corresponding mathematical operations. The result is the following expression for the functional:

$$J = \frac{32}{15}\frac{bhE_1\pi}{l}c_0^2\dot{\sigma}_0\eta\dot{\eta} + \frac{16}{3}bh^2E_1c_0\frac{\pi}{l}\dot{\sigma}_1\dot{\eta} + \frac{16}{15}bhE_1\frac{\pi}{l}c_0^2\dot{\eta}^2\sigma_0 -$$

$$-\frac{bl}{4}E_1^2\dot{\sigma}_0^2\Phi_0 - \frac{l}{h}bE_1^2\dot{\sigma}_0\dot{\sigma}_1\Phi_1 - \frac{bl}{h^2}E_1^2\dot{\sigma}_1^2\Phi_2 + \dot{\eta}c_0\frac{l}{2}. \tag{6}$$

Here

$$\Phi_i = \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} \frac{y^i}{E_{k+1}}dy, \quad i = 0, 1, 2.$$

Thus, the relation (6) is the expression of the functional according to the variational method applied to the solution of the stability problem of a rectangular arch with non-homogeneous thickness.

3 Solution of the problem

In the formula of the functional (6) $\dot{\eta}$, $\dot{\sigma}_0$, $\dot{\sigma}_1$ are independent variable functional arguments. Using the Rayleigh-Ritz method, let's find the stationary values of the functional (6). The condition for the stationarity of the functional (6) is $\delta J = 0$, which means that

$$\frac{\partial J}{\partial \dot{\eta}} = 0, \quad \frac{\partial J}{\partial \dot{\sigma}_0} = 0, \quad \frac{\partial J}{\partial \dot{\sigma}_1} = 0.$$

The last equalities lead to a system of three ordinary differential equations:

$$\begin{gathered}
\frac{64}{15}\frac{h\pi}{l^2}c_0\dot{\sigma}_0\eta + \frac{32}{3}\frac{h^2\pi}{l^2}\dot{\sigma}_1 + \frac{64}{15}\frac{h\pi}{l^2}c_0\dot{\eta}\sigma_0 + \frac{1}{bE_1} = 0, \\
\frac{32}{15}\frac{h\pi}{l}c_0^2\eta\dot{\eta} - \frac{1}{2}lE_1\dot{\sigma}_0\Phi_0 - \frac{l}{h}E_1\dot{\sigma}_1\Phi_1 = 0, \\
\frac{16\pi}{3l}c_0h^2\dot{\eta} - \frac{l}{h}E_1\dot{\sigma}_0\Phi_1 - \frac{2l}{h^2}E_1\dot{\sigma}_1\Phi_2 = 0.
\end{gathered} \tag{7}$$

Let us introduce initial values for η, σ_0, σ_1:

$$\eta(0) = 1, \quad \sigma_0(0) = \sigma_1(0) = 0.$$

Taking into account these conditions and integrating the system, we obtain the following equations:

$$\frac{64}{15}\frac{h\pi}{l^2}c_0\sigma_0\eta+\frac{32}{3}\frac{h^2\pi}{l^2}\sigma_1+\frac{q}{bE_1}=0\,,$$
$$\frac{16}{15}\frac{h\pi}{l}c_0^2\eta^2-\frac{1}{2}lE_1\sigma_0\Phi_0-\frac{l}{h}E_1\sigma_1\Phi_1=\frac{16}{15}\frac{h\pi}{l}c_0^2\,, \tag{8}$$
$$\frac{16\pi}{3l}c_0h^2\eta-\frac{l}{h}E_1\sigma_0\Phi_1-\frac{2l}{h^2}E_1\sigma_1\Phi_2=\frac{16\pi}{3l}c_0h^2\,.$$

For further calculations, are included the following dimensionless quantities:

$$\xi=\frac{c_0}{h}\,,\quad \lambda=\frac{h}{l}\,,\quad \tau=\frac{q}{E_1b}\,,\quad \varphi_0=\frac{E_1}{h}\Phi_0\,,\quad \varphi_1=\frac{E_1}{h^2}\Phi_1\,,\varphi_2=\frac{E_1}{h^3}\Phi_2.$$

Then system (8) in these parameters will take the form:

$$\frac{64\pi}{15}\lambda\xi\sigma_0\eta+\frac{32\pi}{3}\lambda^2\sigma_1+\tau=0\,,$$
$$\frac{16\pi}{15}\xi^2\eta^2-\frac{1}{2}\sigma_0\varphi_0-\varphi_1\sigma_1=\frac{16\pi}{15}\xi^2, \tag{9}$$
$$\frac{16\pi}{3}\xi\lambda\eta-\varphi_1\sigma_0-2\sigma_1\varphi_2=\frac{16\pi}{3}\xi\lambda\,.$$

For further calculations, we present the arch in a three-layer form ($n=3$) and in a periodic structure ($E_1=E_3\,,\ \delta_1=\delta_3$). Then the values of φ_i ($i=0\,,\,1\,,\,2$) will have the following form:

$$\varphi_0=1\,,\quad \varphi_1=0\,,\varphi_2=\frac{1+1,5\beta+0,75\beta^2+0,125\alpha\beta^3}{(1+0,5\beta)^3}\,, \tag{10}$$

where

$$\alpha=\frac{E_1}{E_2}\,,\beta=\frac{\delta_2}{\delta_1}.$$

In system (8), expressing the parameters σ_0 and σ_1 of the series η, substituting in the first equation of the system and considering (10), we find the formula for the dimensionless force:

$$\tau=9,1\pi^2\xi^3\lambda(1-\eta^2)\eta+28,4\pi^2\xi\lambda^3\varphi_2^{-1}(1-\eta). \tag{11}$$

The critical force takes its extreme value under the following condition:

$$\frac{d\tau}{d\eta}=0. \tag{12}$$

Under condition (12), a crack appears in the structures.

4 Numerical experiment

From the equation (12) we find the values η_{cr}:

$$\eta_{cr}=\sqrt{0,33-1,04\frac{\lambda^2}{\xi^2\varphi_2}}\,. \tag{13}$$

Assuming $\xi=10^{-1}\,;\ \lambda=10^{-1}$, we calculate η_{cr}. The corresponding values of η_{cr}, depending on the parameters $\alpha\,,\ \beta$, are given in tables 1 and 2.

Table 1: $\beta = 4$

α	25	35	45	55	65	75
η_{cr}	0.449	0.486	0.506	0.518	0.527	0.534

Table 2: $\alpha = 75$

β	4	9	14	19	24	29
η_{cr}	0.534	0.552	0.556	0.558	0.559	0.560

For all values of the parameters involved in formula (11), the values of the critical force are calculated. Figures 2 and 3 show the dependences of τ_{cr} on the parameters α and β, respectively. As can be seen from figures 2 and 3, as the values of α and β increase, the critical force decreases.

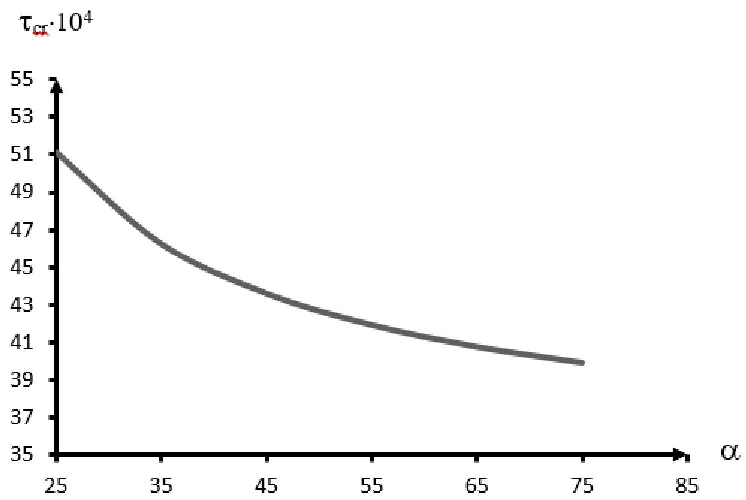

Figure 2: Dependence of the critical force τ_{cr} from the parameter α ($\beta = 4$)

5 Conclusion

The numerical results show:

1. the greatest critical force is achieved at the smallest value of α. The same picture is observed for τ_{cr} and for the parameter β;

2. τ_{cr} will take the maximum value at the smallest values α and β. This means that the geometrical parameter of the middle layer (thickness δ_2) must be less than the geometrical parameter of the outermost layers ($\delta_1 = \delta_3$). In this case, for the middle layer it is necessary to take a harder material than in the extreme layers ($E_2 > E_1 = E_3$).

References

Alfutov, N. (1978). *Fundamentals of Calculating the Stability of Elastic Systems*. Moscow. Engineering, 311p.

Dmitriev, V.G., Egorova, O.V., & Starovoitov, E.I. (2020). Particularities of mathematical modeling of deformation processes for arched and panel designs of composites with large displacements and rotation angles. *INCAS BULLETIN, 12*(Special issue), 53-66.

Dzyuba, A.P., Dzyuba, P.A., & Iskanderov, R.A. (2022). Experimental research of the cylindrical shells stability with rectangular holes of different sizes under transversal bending. *International Journal on Technical and Physical Problems of Engineering*, *14*(3), 94-99.

Emami, E., Kheyroddin, A., & Rezaifar O. (2022). Eccentricity and slenderness ratio effects of arched steel haunches subjected to cyclic loading; experimental study. *Advances in Structural Engineering, 25*(6), 1339–1356.

Fatullayeva, L. (2005). Clicking of non-linear-elastic shallow arch of non-uniform thickness. *Vladikavkaz Mathematical Journal, 7*(2), 86-89.

Fatullayeva, L., Fomina, N., & Mammadova N. (2023). Calculation of critical force during bending of a rectangular arch under pressure. *Advanced Mathematical Models & Applications, 8*(1), 83-91.

Todchuk, V. (2022). On the question of the stability of shells, plates and rods. *Scientific Collection InterConf, 24*(121), 301-324.

Wu, C., Zhang, Q., Huang, W., Zhang, S., & Xu, X. (2022). Cyclic behavior of metallic damper connected reinforced concrete beam-column joint with an opening in slab. *Journal of Building Engineering*, 60, 105193.